Last time:

- encoding a system of equations using an augmented matrix $(A|b)$
- perform row operations to make $(A|b)$ simpler
  - → swap rows
  - → multiply row by $\lambda \neq 0$
  - → add λ times one row to another row
- A is said to be in
  - → echelon form if any all-zero rows of A are at the bottom and any pivot/leading entry of A is to the right of the pivot on the previous row

$$\left(\begin{array}{cccccccc|c} 0&0&0&0&\boxed{7}&\cdots&&&b_1\\ 0&0&0&0&0&0&\boxed{6}&&b_2\\ 0&0&0&0&0&0&0&0&b_3 \end{array}\right)$$

  - → reduced echelon form if above and all pivots are 1 and if any coefficient above any pivot is 0.

$$\left(\begin{array}{cccccccc|c} 0&0&0&0&\boxed{1}&\cdots&0&&b_1\\ 0&0&0&0&0&0&\boxed{1}&&b_2\\ 0&0&0&0&0&0&0&0&b_3 \end{array}\right)$$

- having matrices in REF (reduced echelon form) makes it really easy to solve systems of linear equations

Today: how to "put" any matrix in REF using row operations

Gaussian elimination algorithm below

won't prove

THM 2.1: any matrix can be uniquely put in REF using row operations

Let us illustrate the proof with an example

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 2 & 0 & -2 & 4 \\ 0 & 3 & 0 & 5 & -6 & 2 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

of the algorithm described below

- Beginning: the whole matrix is white
- Middle:

→ shade the all zero column

$\lambda \neq 0$

→ by rescaling row, make $\lambda \to 1$

→ move λ (which has been turned into 1) all the way to top of white part via swaps

→ make all other non-zero entries in column of λ equal to 0 via row operation

→ shade the column of λ & repeat

- End: the whole matrix is shaded in yellow

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 2 & 0 & -2 & 4 \\ 0 & 3 & 0 & 5 & -6 & 2 \end{array}\right) \xrightarrow{\text{shade column}} \left(\begin{array}{ccccc|c} 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & \boxed{2} & 2 & 0 & -2 & 4 \\ 0 & 3 & 0 & 5 & -6 & 2 \end{array}\right) \xrightarrow{\frac{\text{row } 3}{2}}$$

$$\left(\begin{array}{ccccc|c} 0&0&1&-1&1&2 \end{array}\right) \qquad \left(\begin{array}{ccccc|c} 0&0&1&-1&1&2 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 0&0&1&-1&1&2 \\ 0&0&0&1&0&1 \\ 0&\boxed{1}&1&0&-1&2 \\ 0&3&0&5&-6&2 \\ 0&0&1&-1&1&2 \end{array}\right) \xrightarrow{\text{row}1 \leftrightarrow \text{row}3} \left(\begin{array}{ccccc|c} 0&\boxed{1}&1&0&-1&2 \\ 0&0&0&1&0&1 \\ 0&0&1&-1&1&2 \\ 0&3&0&5&-6&2 \\ 0&0&1&-1&1&2 \end{array}\right) \xrightarrow{\text{row }4 - 3\times\text{row}1}$$

then shade

$$\left(\begin{array}{ccccc|c} 0&\boxed{1}&1&0&-1&2 \\ 0&0&0&1&0&1 \\ 0&0&\boxed{1}&-1&1&2 \\ 0&0&-3&5&-3&-4 \\ 0&0&1&-1&1&2 \end{array}\right) \xrightarrow{\text{row }2 \leftrightarrow \text{row }3} \left(\begin{array}{ccccc|c} 0&\boxed{1}&1&0&-1&2 \\ 0&0&\boxed{1}&-1&1&2 \\ 0&0&0&1&0&1 \\ 0&0&-3&5&-3&-4 \\ 0&0&1&-1&1&2 \end{array}\right) \xrightarrow[\text{row}5-\text{row}2]{\text{row }1-\text{row}2,\ \text{row}4+3\text{row}2}$$

then shade

$$\left(\begin{array}{ccccc|c} 0&\boxed{1}&0&1&-2&0 \\ 0&0&\boxed{1}&-1&1&2 \\ 0&0&0&\boxed{1}&0&1 \\ 0&0&0&2&0&2 \\ 0&0&0&0&0&0 \end{array}\right) \xrightarrow[\text{row }4-2\text{row}3]{\text{row }1-\text{row }3,\ \text{row }2+\text{row}3} \left(\begin{array}{ccccc|c} 0&\boxed{1}&0&0&-2&-1 \\ 0&0&\boxed{1}&0&1&3 \\ 0&0&0&\boxed{1}&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{array}\right)$$

then shade

free columns (1,5)

$$\left(\begin{array}{ccccc|c} 0&\boxed{1}&0&0&-2&-1 \\ 0&0&\boxed{1}&0&1&3 \\ 0&0&0&\boxed{1}&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{array}\right)$$

is in REF

(reduced echelon form)

pivot columns (2,3,4)

pivot variables : x_2, x_3, x_4

(basic variables)

free variables : x_1, x_5

$\qquad\qquad\qquad\quad\ \ \| \quad \|$

$\qquad\qquad\qquad\quad\ \ s \ ,\ t \in \mathbb{R}$

system
$$\begin{cases} x_2 \qquad\qquad -2x_5 = -1 \\ \qquad x_3 \qquad + x_5 = 3 \\ \qquad\qquad x_4 \qquad = 1 \\ \qquad\qquad\qquad 0 = 0 \end{cases}$$

$$\Longrightarrow \begin{array}{l} x_2 = 2x_5 - 1 = 2t-1 \\ x_3 = 3 - x_5 = 3 - t \end{array}$$

$0 = 0$ $\qquad x_4 = 1$

$\Rightarrow$ solution is
$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} s \\ 2t-1 \\ 3-t \\ 1 \\ t \end{pmatrix}, \ \underbrace{\forall}_{\text{"for any"}} \ s,t \in \mathbb{R} \right\}$$

The row echelon form of $(A|b)$ gives us information about the number of solutions of the associated system

- when the REF has a row $0\,0\,0\,0\,0\,0\,0\,0 \,|\, \underbrace{b}_{\neq 0}$
  then $\underbrace{\exists}_{\text{"exists"}}$ 0 solutions
- if any all zero row of the REF looks like $0\,0\,0\,0\,0\,0\,0 | 0$, then $\exists \geq 1$ solution
  - if there are no free variables, $\exists$ 1 solution
  - if there are free variables, $\exists\ \infty$ solutions

(if $\exists$ k free variables, this infinity is k-dimensional)

Fact: (# of free variables) + (# of pivot variables) $= n$

columns of A
variables

Vectors (in $\mathbb{R}^n$ = n-dimensional space)

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1, x_2, \dots, x_n \in \mathbb{R} \right\}$$

Operations : Addition : (subtraction)

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \pm \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{pmatrix}$$

only vectors of the same size can be added.

$$0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + 0 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad -\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{pmatrix}$$

zero vector

Scalar multiplication : $\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}$

number (scalar)

scalar vector

$$0 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \qquad 1 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad (\lambda \cdot \mu) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \cdot \left(\mu \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right)$$

Rules:
- vector + vector = vector
- scalar • vector = vector

Ex : $2 \cdot \underbrace{\begin{pmatrix} 3 \\ 7 \\ -5 \end{pmatrix}}_{v_1} + 5 \cdot \underbrace{\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}}_{v_2} = \begin{pmatrix} 6 \\ 14 \\ -10 \end{pmatrix} + \begin{pmatrix} -5 \\ 0 \\ 20 \end{pmatrix} = \begin{pmatrix} 1 \\ 14 \\ 10 \end{pmatrix}$

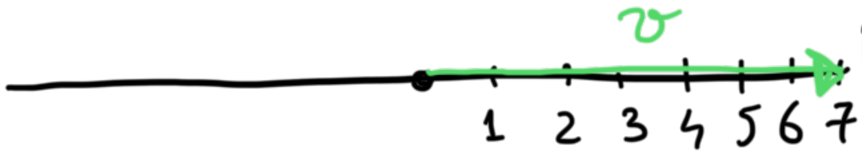
DEF 2.2: a linear combination of vectors $v_1, \dots, v_m \in \mathbb{R}^n$

is any vector of the form

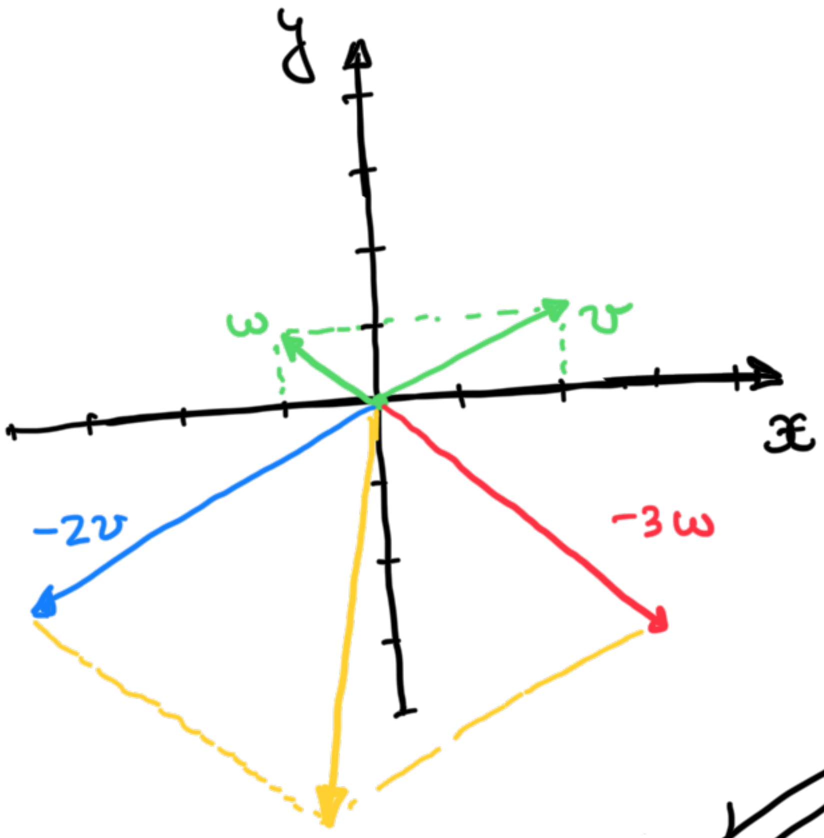
$$\boxed{c_1 v_1 + c_2 v_2 + \ldots + c_m v_m}$$

where $c_1, c_2, \ldots, c_m$ are arbitrary real numbers

Geometry:



$\mathbb{R}^1 \ni v = \begin{pmatrix} 7 \end{pmatrix}$



$\mathbb{R}^2 \ni v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\ni w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

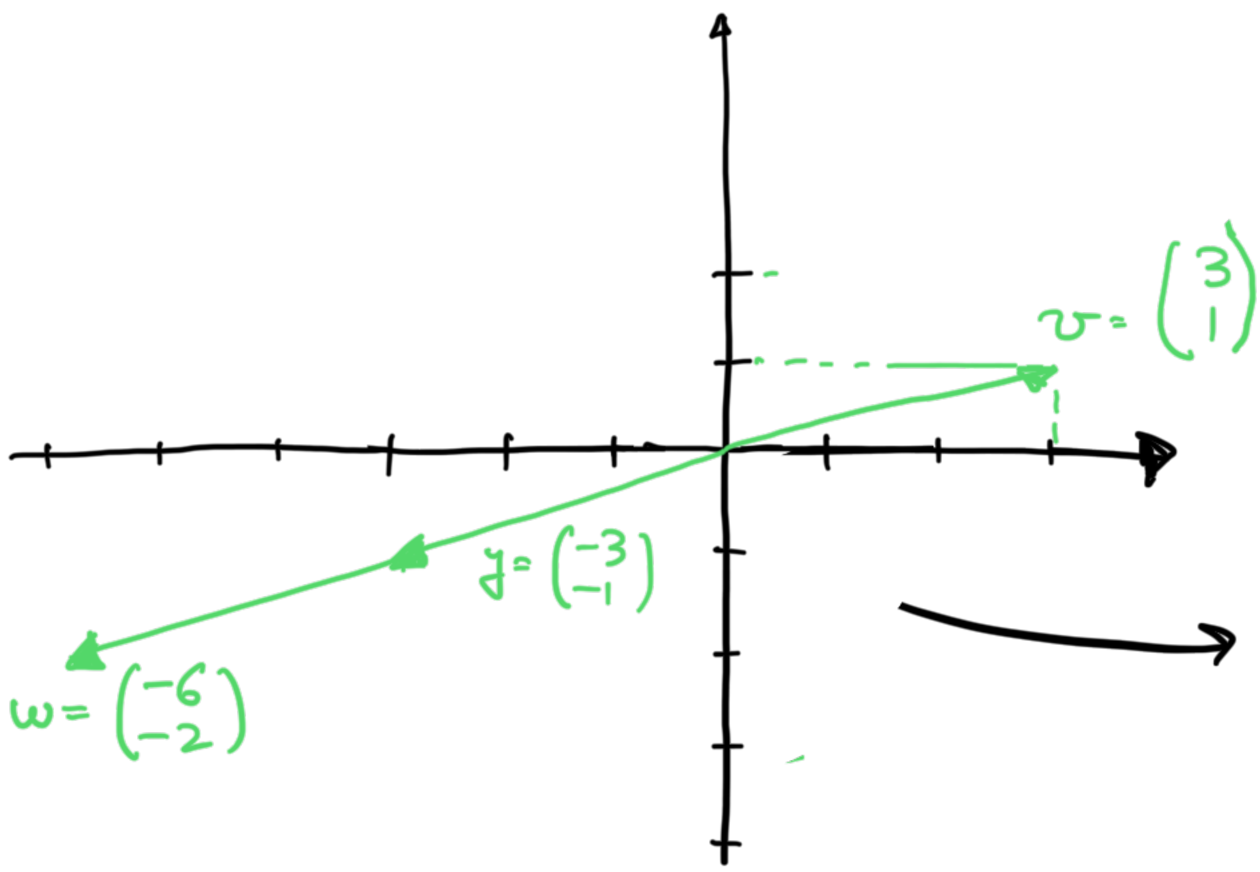
$-2v = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$

$-3w = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

$$-2v - 3w = (-2)v + (-3)w = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$$

Above, any vector in $\mathbb{R}^2$ is a linear combination of v and w

$\Rightarrow \mathbb{R}^2 = \text{span}\{v, w\}$



$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Linear combinations of v, w and y are constrained to the green line $\Rightarrow$ Span(v, w, y) = green line

DEF 2.3: the span of vectors $v_1, \ldots, v_m \in \mathbb{R}^n$ is the set of all possible linear combinations of $v_1, \ldots, v_m$

Denoted Span$\{v_1, \ldots, v_m\}$

can be $\bullet$, can be a line, can be a plane, ...

Question: how can you tell if a given vector $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is a linear combination of given vectors

$$v_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \ldots\ldots\ldots, v_m = \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{nm} \end{pmatrix} ?$$

such that

i.e. when $\exists\ c_1, \ldots, c_m \in \mathbb{R}$ s.t. $c_1 v_1 + \ldots + c_m v_m = b$?

$$\begin{cases} c_1 a_{11} + \ldots\ldots + c_m a_{1m} = b_1 \\ c_1 a_{21} + \ldots\ldots + c_m a_{2m} = b_2 \\ \vdots \\ c_1 a_{n1} + \ldots\ldots + c_m a_{nm} = b_n \end{cases}$$

Answer: precisely when the above system of eqns has a solution